



TITLE:

Research problems in number theory III.(Analytic Number Theory and Surrounding Areas)

AUTHOR(S):

Katai, I.

CITATION:

Katai, I.. Research problems in number theory III.(Analytic Number Theory and Surrounding Areas). 数理解析研究所講究録 2006, 1511: 121-128

ISSUE DATE:

2006-08

URL:

<http://hdl.handle.net/2433/58603>

RIGHT:

Research problems in number theory III.

By
I. Kátai

Abstract

Some open problems in the field of arithmetic functions are presented.

Research problems in number theory III.

By
I. Kátai*

§1. Introduction

In this paper we shall formulate some open problems, conjectures in the field of number theory. Some of them were formulated earlier in one of my papers [1], [2], [3], [4].

Notations. As usual $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, integers, rational, real, or complex numbers, respectively. Let $\mathbb{Q}_x, \mathbb{R}_x$ be the multiplicative group of positive elements of \mathbb{Q}, \mathbb{R} , resp. Let \mathcal{P} be the set of prime numbers. Let $P(n)$ be the largest prime factor of n .

§2. Continuous homomorphisms as arithmetical functions

2.1. For some additively written commutative group G let \mathcal{A}_G^* be the set of those functions $f : \mathbb{N} \rightarrow G$, for which $f(mn) = f(m) + f(n)$ holds for every couple of $m, n \in \mathbb{N}$.

We say that \mathcal{A}_G^* is the class of completely additive functions.

Let G be multiplicatively written, commutative group, \mathcal{M}_G^* be the class of those $g : \mathbb{N} \rightarrow G$, for which $g(mn) = g(m) \cdot g(n)$ for every pair of $m, n \in \mathbb{N}$. We say that \mathcal{M}_G^* is the set of completely multiplicative functions.

If $f \in \mathcal{A}_G^*$, then its domain \mathbb{N} can be extended to \mathbb{Q}_x by

$$f\left(\frac{m}{n}\right) := f(m) - f(n),$$

and the equation

$$f(r_1 r_2) = f(r_1) + f(r_2)$$

remains valid for every $r_1, r_2 \in \mathbb{Q}_x$.

Let G be a topological group and $f : \mathbb{Q}_x \rightarrow G$, $f \in \mathcal{A}_G^*$ be continuous at 1. Then, for each $\alpha \in \mathbb{R}_x$ there exists the limit

$$\lim_{r \rightarrow \alpha} f(r) =: \Phi(\alpha),$$

Φ is continuous everywhere in \mathbb{R}_x , furthermore $\Phi(\alpha\beta) = \Phi(\alpha) + \Phi(\beta)$ is valid for all $\alpha, \beta \in \mathbb{R}_x$, i.e. ϕ is a continuous homomorphism of \mathbb{R}_x into G .

The following conjectures 1,2 are proposed by M.V. Subbarao and myself.

Conjecture 1. Let G be a compact Abelian topological group, $f \in \mathcal{A}_G^*$, and let the closure of $f(\mathbb{N})$ be G (closure $f(\mathbb{N})$ is a closed subgroup in G). Let U be the set of those u for which there exists an infinite sequence of integers $n_\nu \nearrow$, such that $f(n_\nu + 1) - f(n_\nu) \rightarrow u$.

Then U is a subspace in G , furthermore $f(n) := \Phi(n) + V(n)$, where Φ is a continuous homomorphism, $\phi : \mathbb{R}_x \rightarrow G$, $V(\mathbb{N}) \subseteq U$, $\text{clos } V(\mathbb{N}) = U$.

The next conjecture is a special case of Conjecture 1.

Conjecture 2. Let $f : \mathbb{N} \rightarrow \mathbb{C}$ be a completely multiplicative function, $|f(n)| = 1$ ($n \in \mathbb{N}$), $\delta_f(n) = f(n+1)\bar{f}(n)$.

Let $\mathcal{A}_k = \{\alpha_1, \dots, \alpha_k\}$ be the set of limit points of $\{\delta_f(n) \mid n \in \mathbb{N}\}$. Then $\mathcal{A}_k = S_k$, where S_k is the set of k 'th complex units, i.e. $S_k = \{w \mid w^k = 1\}$, furthermore $f(n) = n^{i\tau} F(n)$ with a suitable $\tau \in \mathbb{R}$, and $F(\mathbb{N}) = S_k$, and for every $w \in S_k$ there exists a sequence $n_\nu \nearrow \infty$ such that $F(n_\nu + 1)\bar{F}(n_\nu) = w$ ($\nu = 1, 2, \dots$).

The motivation of the problems, and partial results can be read in [7], [8].

*Research supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and by a grant from OTKA T46993.

A weakened form of Conjecture 2 has been proved by E. Wirsing recently [6]: Under the conditions of Conjecture 2, $\mathcal{A}_k \subseteq S_l$ for a suitable l , and $f(n) = n^{it} F(n)$, $\mathbb{F}(\mathbb{N}) \subseteq S_l$.

2.2. Let $T = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus. Let \mathcal{A}_T be the set of additive functions taking values from T , i.e. $F: \mathbb{N} \rightarrow T$ belongs to \mathcal{A}_T if $F(mn) = F(m) + F(n)$ holds for all coprime pairs of m, n . We say that F is of finite support, if $F(p^\alpha) = 0$ holds for every large prime p , and every $\alpha \in \mathbb{N}$.

For $F_\nu \in \mathcal{A}_T$ ($\nu = 0, \dots, k-1$) let

$$L_n(F_0, \dots, F_{k-1}) := F_0(n) + \dots + F_{k-1}(n+k-1).$$

Let $\mathcal{L}_0^{(k)}$ be the space of those k -tuples (F_0, \dots, F_{k-1}) of $F_\nu \in \mathcal{A}_T$ for which

$$L_n(F_0, \dots, F_{k-1}) = 0 \quad (n \in \mathbb{N})$$

holds.

Conjecture 3. $\mathcal{L}_0^{(k)}$ is a finite dimensional \mathbb{Z} module, and each F_j is of finite support.

Conjecture 4. If $F_\nu \in \mathcal{A}_T$ ($\nu = 0, 1, \dots, k-1$), and

$$L_n(F_0, \dots, F_{k-1}) \rightarrow 0 \quad (n \rightarrow \infty)$$

then there exist suitable real numbers $\tau_0, \dots, \tau_{k-1}$ such that $\tau_0 + \dots + \tau_{k-1} = 0$, and for $H_j(n) := F_j(n) - \tau_j \log n$ ($j = 0, \dots, k-1$) we have

$$L_n(H_0, \dots, H_{k-1}) = 0 \quad (n = 1, 2, \dots).$$

Conjecture 5. For every integer $k(\geq 1)$ there exists a constant c_k such that for every prime p greater than c_k ,

$$\min_{j=1, \dots, p-1} \max_{\substack{l \in [-k, k] \\ l \neq 0}} P(jp + l) < p.$$

The conjecture is open even in the case $k = 2$.

Let Q_x^l be the l -fold direct product of Q_x . Let furthermore O_l be its subgroup, generated by the elements $(n, n+1, \dots, n+l-1)$ ($n \in \mathbb{N}$).

The following assertions are true:

(1) Let $\mathcal{L}_0^{*(l)}$ be the space of those l -tuples (F_0, \dots, F_{l-1}) of $F_\nu \in \mathcal{A}_T^*$ for which $L_n(F_0, \dots, F_{l-1}) = 0$ ($n \in \mathbb{N}$). Assume that Conjecture 5 is true for $k = l$. Then $\mathcal{L}_0^{*(l)}$ is a finite dimensional space.

(2) $\mathcal{L}_0^{*(l)}$ (defined in (1)) is of finite dimensional, if and only if the factor group Q_x^l/O_l is finite. $\mathcal{L}_0^{*(l)}$ is trivial (it contains only $(0, \dots, 0)$) if and only if $O_l = Q_x^l$.

2.3. Let $\mathcal{A}^* = \mathcal{A}_{\mathbb{R}}^*$.

Definition 1. (Set of uniqueness). We say that $E \subseteq \mathbb{N}$ is a set of uniqueness for the class of completely additive functions, if $f \in \mathcal{A}^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

Definition 2. (Set of uniqueness mod 1). We say that $E \subseteq \mathbb{N}$ is a set of uniqueness mod 1, if $f \in \mathcal{A}_T^*$, $f(E) = 0$ implies that $f(\mathbb{N}) = 0$.

I introduced the notion "set of uniqueness" in [10] and proved [11] that the set of "primes + 1" can be extended by finitely many integers so that the resulting set is a set of uniqueness. My guess that the set of shifted primes itself is a set of uniqueness, was proved by Elliott [12]. It was proved by Wolke [13] that E is a set of uniqueness if and only if for every $n \in \mathbb{N}$ there exists a suitable $k \in \mathbb{N}$, such that

$$n^k = \prod_{i=1}^h a_i^{\varepsilon_i}, \quad \text{where } a_i \in E, \varepsilon_i = \pm 1.$$

It was proved (by Meyer, Indlekofer, Dress and Volkman, Hoffman, Elliott, independently) that E is a set of uniqueness mod 1, if every $n \in \mathbb{N}$ can be written as

$$n = \prod_{j=1}^s a_j^{d_j}, \quad a_j \in E, \quad d_j \in \mathbb{Z}, \quad (j = 1, \dots, s).$$

Conjecture 6. *The set of "prime + 1" s is a set of uniqueness mod 1.*

Conjecture 6 is proposed by several mathematicians independently.

A quite detailed treatment of this topic is given by Elliott [14].

Indlekofer and Timofeev proved that $\{u^2 + v^2 + a \mid u, v \in \mathbb{Z}\}$ is a set of uniqueness mod 1, if $a \neq 0$.

The same result is proved by De Koninck and Kátai.

§3. On q -additive and q -multiplicative functions

Let $q \geq 2$ be an arbitrary integer, $\mathcal{E} = \{0, 1, \dots, q-1\}$ and let $\varepsilon_0(n), \varepsilon_1(n), \dots$ be the digits in the q -ary expansion of $n : n = \varepsilon_0(n) + \varepsilon_1(n)q + \dots$. This is a finite expansion, since $\varepsilon_j(n) = 0$ if $q^j > n$.

Let $f : \mathbb{N}_0 \rightarrow \mathbb{R}$ be such a function for which $f(0) = 0$, and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j)$$

holds for every n . We say that f is q -additive, and the set of q -additive functions is denoted by \mathcal{A}_q .

We say that $g : \mathbb{N}_0 \rightarrow \mathbb{C}$ is q -multiplicative if $g(0) = 1$, and $g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j)$ holds for every

n . Let \mathcal{M}_q be the set of q -multiplicative functions, and $\overline{\mathcal{M}}_q$ be those of \mathcal{M}_q for the elements $g \in \overline{\mathcal{M}}_q$ additionally $|g(n)| = 1$ ($n \in \mathbb{N}_0$) holds as well.

Let $g \in \overline{\mathcal{M}}_q$,

$$P(x) = \sum_{p \leq x} g(p), \quad S(x \mid \alpha) = \sum_{\substack{l < x \\ (l, q) = 1}} g(l)e(\alpha l)$$

where $e(y) := e^{2\pi i y}$.

We are interested in to give necessary and sufficient conditions for g to satisfy

$$(3.1) \quad \lim_{x \rightarrow \infty} \frac{P(x)}{\pi(x)} = 0.$$

Conjecture 7. *Let $g \in \overline{\mathcal{M}}_q$. Then (3.1) holds if and only if*

$$(3.2) \quad x^{-1}S(x, r) \rightarrow 0$$

for every $r \in \mathbb{Q}$.

The necessity of (7.2) is quite obvious, since if it does not hold, then

$$\sum_{j=0}^{\infty} \sum_{c \in E} \operatorname{Re} (1 - g(cq^j)e(cq^j r)) < \infty,$$

whence one can prove easily that (3.2) cannot hold. The difficulty is in the sufficiency.

Let $T_{l_1, l_2}^M = T_{l_1, l_2} =$

$$= \#\{p_1, p_2 \in \mathcal{P}, p_2 - p_1 = l_2 - l_1, p_1 \equiv l_1 \pmod{q^M}, p_2 \leq x\},$$

$$H(d) := \prod_{\substack{p|d \\ p \nmid 2q}} \left(1 + \frac{1}{p-2}\right).$$

Conjecture 8. *There exists a constant $\delta \in (0, 1/2)$, such that for $M = [\delta N]$, $N = \left\lceil \frac{\log x}{\log q} \right\rceil$,*

$$\sum_{\substack{l_1, l_2 < q^M \\ (l_1 l_2, q) = 1 \\ l_1 \neq l_2}} \left| T_{l_1, l_2}^{(M)} - \frac{x}{\varphi(q^M)(\log x)^2} H(l_2 - l_1) \right| < \frac{\varepsilon(x)x \cdot q^M}{(\log x)^2}$$

with a suitable function $\varepsilon(x) \rightarrow 0$ ($x \rightarrow \infty$).

In [15] we proved that Conjecture 8 implies the fulfilment of Conjecture 1.

Furthermore in [15] we proved the following assertion: Let $Y(x) \nearrow \infty$, so that $\frac{\log Y(x)}{\log x} \rightarrow 0$. Let $\mathcal{N}_x := \{n \leq x, p(n) > Y(x)\}$, where $p(n)$ is the smallest prime factor of n .

Let $N(x) = \text{card } (\mathcal{N}_x)$. Let $L(x)$ be strongly multiplicative, $(L(p^h) =) L(p) = \frac{1}{p-2}$ if $p \nmid 2q$, and $L(p) = 0$ otherwise. Let

$$U(x) := \sum_{n \in \mathcal{N}_x} g(n).$$

Then

$$\left| \frac{U(x)}{N(x)} \right|^2 \leq \sum_{d < D} \frac{L(d)}{d} \sum_{a=0}^{d-1} \left| q^{-M} S \left(q^M \mid \frac{a}{d} \right) \right|^2 + \frac{c_1}{D} + o_x(1),$$

where $M = \left\lceil \frac{1 \log x}{4 \log q} \right\rceil$, c_1 is a positive constant which depends only on q , $o_x(1)$ does depend on $Y(x)$, and D is an arbitrary real numbers.

§4. The distribution of q -ary digits on some subsets of integers

4.1. Let $\mathcal{B}(\subseteq \mathbb{N}_0)$ be infinite, $B(x) = \#\{b \leq x, b \in \mathcal{B}\}$. For $0 \leq l_1 < l_2 < \dots < l_h$, $b_1, \dots, b_h \in E$, let $A_{\mathcal{B}} \left(x \mid \begin{smallmatrix} l \\ b \end{smallmatrix} \right)$ be the size of those integers $n \in \mathcal{B}$, $n \leq x$, for which $\varepsilon_{l_j}(n) = b_j$ ($j = 1, \dots, h$) simultaneously hold.

Conjecture 9. *For every $h \left(\leq \frac{N}{3} \right)$, $1 \leq l_1 < \dots < l_h (\leq N)$, and $b_1, \dots, b_h \in \mathcal{E}$ denote*

$$\left(\Delta_h \left(\begin{smallmatrix} l \\ b \end{smallmatrix} \right) = \right) \Delta_h \left(\begin{smallmatrix} l_1, \dots, l_h \\ b_1, \dots, b_h \end{smallmatrix} \right) := \frac{q^h A_{\mathcal{P}} \left(q^n \mid \begin{smallmatrix} l \\ b \end{smallmatrix} \right)}{\pi(q^N)} - 1.$$

Then

$$(4.1) \quad \sup_{1 \leq h \leq \frac{N}{3}} \sup_{\substack{l_1, \dots, l_h \\ b_1, \dots, b_h}} \left| \Delta_h \left(\begin{smallmatrix} l \\ b \end{smallmatrix} \right) \right| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Here \mathcal{P} is the set of primes.

Remarks. 1. Inequality, similar but much weaker than (4.1) was proved in [16].

2. These type of inequalities would be interesting for other sets \mathcal{B} instead of \mathcal{P} , like $\mathcal{B} = \{\text{fixed polynomial } (n) \mid n \in \mathbb{N}\}$, or $= \{\text{fixed polynomial } (p) \mid p \in \mathcal{P}\}$. We would be able to use them in proving

central limit theorems with remainder terms for $f(P(n))$, or $f(P(p))$, where $f \in \mathcal{A}_q$, $P = \text{polynomial}$. (See [17], [18], [19], [20], [21]).

4.2.

Conjecture 10. If $g \in \overline{\mathcal{M}}_q$, $g(p) = 1$ for every $p \in \mathcal{P}$, then $g(nq) = 1$ ($n \in \mathbb{N}$).

See [22], where it is proved that there exists an absolute constant $c(>0)$ such that $g \in \overline{\mathcal{M}}_q$, $g(p) = 1$ implies that there exists an integer k , $1 \leq k \leq c$ for which $g^k(nq) = 1$ ($n \in \mathbb{N}$).

§5. On a theorem of H. Daboussi

H. Daboussi proved several years ago that for $f \in \mathcal{A}$, $|f(n)| \leq 1$, and for every irrational α , in the notation

$$m(f, \alpha, x) := \frac{1}{x} \left| \sum_{n \leq x} f(n) e(n\alpha) \right|,$$

we have

$$\sup_{f \in \mathcal{A}, |f| \leq 1} m(f, \alpha, x) \rightarrow 0 \quad (x \rightarrow \infty).$$

This theorem has been generalized in different directions.

Let \mathcal{P}_k be the set of square-free numbers n with exactly k prime-factors: $n = p_1 p_2 \dots p_k$. Let α be an irrational number. Let $q_1 < q_2 < \dots < q_r$ be the whole set of primes less than x . Let X_{q_j} ($j = 1, \dots, r$) be complex numbers,

$$Q_k(X_{q_1}, \dots, X_{q_r}) := \left| \sum_{\substack{n=p_1 \dots p_k < x \\ n \in \mathcal{P}_k}} X_{p_1} \dots X_{p_k} e(n\alpha) \right|.$$

Let

$$\delta_k(x) := \max_{|X_{q_1}| \leq 1, \dots, |X_{q_r}| \leq 1} \frac{Q_k(X_{q_1}, \dots, X_{q_r})}{\pi_k(x)},$$

$$\delta_k := \limsup_{x \rightarrow \infty} \delta_k(x),$$

where $\pi_k(x)$ is the number of $n \leq x$, $n \in \mathcal{P}_k$.

Conjecture 11. We have $\delta_k < 1$, if $k \geq 2$. Furthermore $\delta_k \rightarrow 0$ (if $k \rightarrow \infty$).

Remark. Recently I could prove that $\delta_2 = 0$ for almost all α .

§6. Some problems originated from Rényi-Parry expansions

See our papers written jointly with Daróczy [23 - 26].

Let \mathbb{C}^∞ denote the space of sequences $\underline{c} = (c_0, c_1, \dots)$ the coordinates c_ν of which $\in \mathbb{C}$. This shift operator $\sigma : \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$ is defined by $\sigma((c_0, c_1, \dots)) = (c_1, c_2, \dots)$. Let $t_0 = 1$, $t_\nu \in \mathbb{C}$, t_ν be bounded, $\underline{t} = (t_0, t_1, \dots)$. Let

$$(6.1) \quad R(z) = t_0 + t_1 z + \dots$$

Let l_1 be the linear space of those $\underline{b} \in \mathbb{C}^\infty$, for which $\sum |b_\nu| < \infty$.

The scalar product of an element $\underline{b} \in l_1$ and a bounded sequence \underline{c} let:

$$\underline{c} \underline{b} = \underline{b} \underline{c} = \sum_{\nu=0}^{\infty} b_\nu c_\nu.$$

Let

$$(6.2) \quad \mathcal{H}_t := \{\underline{b} \in l_1 \mid \sigma^l(\underline{b})\underline{t} = 0, l = 0, 1, 2, \dots\}.$$

It is clear that \mathcal{H}_t is a closed linear subspace of l_1 , furthermore $\sigma(\mathcal{H}_t) \subseteq \mathcal{H}_t$.

Let $\mathcal{H}_t^{(0)} \subseteq \mathcal{H}_t$ be the set of those $\underline{b} \in \mathcal{H}_t$ for which

$$(6.3) \quad |b_\nu| \leq C(\varepsilon, \underline{b})e^{-\varepsilon\nu} \quad (\nu \geq 0)$$

holds with some $\varepsilon > 0$ and finite $C(\varepsilon, \underline{b})$.

If ρ is a root of $R(z)$, $|\rho| < 1$, then $b_\nu := \rho^\nu$ satisfies $\sigma^l(\underline{b})\underline{t} = 0$ ($l = 0, 1, \dots$), and $|b_\nu| \leq C \cdot e^{-\varepsilon\nu}$ with $C = 1$, and with ε counted from $e^{-\varepsilon} = |\rho|$. If the order of the multiplicity of the root ρ is m , then $\underline{b} \in \mathcal{H}_t$, if $b_\nu = \nu^j \rho^\nu$ ($\nu = 0, 1, \dots$), for every $j = 0, 1, \dots, m-1$. The sequences $b_\nu = \nu^j \rho^\nu$ ($\nu = 0, 1, \dots$) are called elementary solutions. Let $\mathcal{H}_t^{(e)}$ be the space of finite linear combinations of the elementary solutions, and let $\overline{\mathcal{H}_t^{(e)}}$ be the closure of $\mathcal{H}_t^{(e)}$.

Conjecture 12. We have: $\overline{\mathcal{H}_t^{(e)}} = \mathcal{H}_t$.

Conjecture 13. Assume that $R(z) \neq 0$ in $|z| < 1$. Then $\mathcal{H}_t = \{0\}$.

References

- [1] I. Kátai, *Research problems in number theory I*, Matematikai Lapok **19** (1968), 317-325. (Hungarian)
- [2] I. Kátai, *Research problems in number theory*, Publicationes Math. Debrecen **24** (1977), 263-276.
- [3] I. Kátai, *Research problems in number theory II*, Annales Univ. Eötvös Loránd (Budapest), Sectio Computatorica **16** (1996), 223-251.
- [4] I. Kátai, *Characterization of arithmetical functions, problems and results*, Number Theory, Proc. of the International Number Theory Conference held at Université Laval, 1987, W. de Gruyter, New York 1989, pp. 544-555.
- [5] I. Kátai, *Generalized number systems in Euclidean spaces*, Mathematical and Computer Modelling **38** (2003), 883-892.
- [6] E. Wirsing, *On a problem of Kátai and Subbarao*, Annales Univ. Eötvös Loránd (Budapest), Sectio Computatorica. (accepted)
- [7] I. Kátai and M.V. Subbarao, *The characterization of n^{it} as a multiplicative function*, Acta Math. Hung.
- [8] I. Kátai and M.V. Subbarao, *On the multiplicative function n^{it}* , Studia Sci. Math. **34** (1998), 211-218.
- [9] I. Kátai, *Continuous homomorphisms as arithmetical functions, and sets of uniqueness*, Number Theory, edited: R.P. Bambah, V.C. Dumir, R.J. Hans-Gill, Hindustan Book Agency and Indian National Science Academy, 2000, pp. 183-200.
- [10] I. Kátai, *On sets characterizing number - theoretical functions*, Acta Arithm. **13** (1968), 315-320.
- [11] I. Kátai, *On sets characterizing number - theoretical functions II. (The set of "prime plus one"s is a set of quasi uniqueness)*, Acta Arithm. **26** (1974), 11-20.
- [12] P.D.T.A. Elliott, *On two conjectures of Kátai*, Acta Arithm. **26** (1974), 11-20.
- [13] D. Wolke, *Bemerkungen über Eindeutigkeits - mengen additiver Funktionen*, Elem. der Math. **33** (1978), 14-16.
- [14] P.D.T.A. Elliott, *Arithmetic functions and integer products*, Springer V., New York 1985.
- [15] I. Kátai, *Distribution of q -additive functions*, Probability theory and applications, eds. J. Galambos and I. Kátai, Kluwer, 1992, 309-318.
- [16] I. Kátai, *Distribution of digits of primes in q -ary canonical form*, Acta Math. Hung. **47** (1986), 341-359.
- [17] N.L. Bassily and I. Kátai, *Distribution of the values of q -additive functions on polynomial sequences*, Acta Math. Hung. **68**(4), (1995), 353-361.
- [18] N.L. Bassily and I. Kátai, *Distribution of consecutive digits in the q -ary expansions of some subsequences of integers*, Journal of Mathematical Sciences **78**(1), (1996), 11-17.

- [19] M. Drmota, *The joint distribution of q -additive functions*, Acta Arithm. **100**(1), (2001), 17-39.
- [20] B. Gittenberger and J.M. Thuswaldner, *Asymptotic normality of b -additive functions on polynomial sequences in the Gaussian number field*, J. Number Theory **84** (2000), 317-341.
- [21] W. Steiner, *The distribution of digital expansions on polynomial sequences (Dissertation)*, Technische Universität Wien, Institut für Geometrie, 2002.
- [22] K.-H. Indlekofer and I. Kátai, *On q -multiplicative functions taking a fixed value on the set of primes*, Studia
- [23] Z. Daróczy, A. Járαι, I. Kátai, *Intervallfüllende Folgen und volladditive Funktionen*, Acta Sci. Math. (Szeged), **50** (1986), 337-350.
- [24] Z. Daróczy and I. Kátai, *Additive functions*, Analysis Math. **12** (1986), 85-96.
- [25] Z. Daróczy and I. Kátai, *Interval filling sequences and additive functions*, Acta Sci. Math. (Szeged), **52** (1988), 337-347.
- [26] Z. Daróczy and I. Kátai, *Continuous additive functions and difference equations of infinite order*, Analysis Math. **12** (1986), 237-249.